

Investigation of a Toy Model for Frustration in Abelian Lattice Gauge Theory

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(February 7, 2008)*

We introduce a lattice model with local $U(1)$ gauge symmetry which incorporates explicit frustration in $d > 2$. The form of the action is inspired from the loop expansion of the fermionic determinant in standard lattice QED. We study through numerical simulations the phase diagram of the model, revealing the existence of a frustrated (antiferromagnetic) phase for $d = 3$ and $d = 4$, once an appropriate order parameter is identified.

I. INTRODUCTION

The importance of frustration and disorder is well known to condensed matter physicists working in the field of spin-glasses and related systems [1]. In these systems one can find a variety of unusual phases which, in some cases, present completely new phenomena as, for instance, the existence of a multiplicity of vacua not related by a symmetry of the action. It is interesting to ask whether similar phenomena could happen in a field theory and what will be the physical consequences.

We address here the issue of frustration in lattice gauge theories. The motivation for this work relies on the fact that, as well known, realistic gauge theories with dynamical fermions, in contrast to gauge theories with fundamental scalar fields, are characterized by a frustrated effective action. Frustration here must be understood in the most naive way, i.e., after the integration of the Grassmann degrees of freedom, the effective fermion action has contributions which can be seen as competing interactions.

Even if spin-glass like phases in gauge theories have been found time ago we are interested, as stated before, in more realistic gauge theories. More precisely, strongly coupled QED in the noncompact lattice regularization has received considerable attention in recent time ([2,3] and references therein). Many interesting and exciting

new results in the field of four-dimensional gauge theories, as for instance the possibility to have a non Gaussian fixed point [3], appear in this model. The origin of these new phenomena is unfortunately not clear at all. It has been argued in the past that the monopole percolation transition could drive the chiral transition of strongly coupled QED [4]. This issue is however rather unclear and it is therefore worthwhile to explore other possibilities. There are furthermore other exciting results in non compact QED as the existence of a non chiral phase transition in the three-dimensional model [5] or the fact that the transition seems to prolongate to non vanishing fermion masses [6] which allow one to speculate that frustration could be the origin of these phenomena. We should say from the beginning that at the moment this is only speculation. Our aim is to investigate this issue and, as a first step in this direction, we will study in this paper a model [7] in which frustration is introduced by hand, although inspiration is taken from the hopping-parameter expansion of QED. Section II is devoted to review how frustration appears in lattice QED. We analyze this phenomenon from the hopping parameter expansion as well as from the loop expansion of the determinant of the Dirac-Kogut-Susskind operator. In Sec. III we define our toy model, discuss its classical structure of ground states, introduce an order parameter for the phase transition and show how our model in two dimensions is equivalent to the XY-spin model. Section IV contains the numerical results of a simulation and the phase structure. The conclusions are reported in Sec. V.

II. FRUSTRATION IN LATTICE QED

We start with the action functional for QED on the lattice using Kogut-Susskind fermions, $S = \beta S_G + S_F$, with S_G the pure gauge action (either compact or non-compact) and S_F the fermionic action,

$$S_F = m \sum_x \bar{\chi}(x) \chi(x) + \frac{1}{2} \sum_{x, \mu} \eta_\mu(x) \bar{\chi}(x) \{U_\mu(x) \chi(x + \mu) - U_\mu^*(x - \mu) \chi(x - \mu)\} \quad (1)$$

where x labels the points and μ the forward unit vectors in the lattice. The $U_\mu(x)$ are compact $U(1)$ link variables, $\chi(x)$ are fermion fields (Grassmann variables) and $\eta_\mu(x)$ are the Kogut-Susskind phases, defined as usual,

$$\eta_\mu(x) = (-1)^{x_1 + x_2 + \dots + x_{\mu-1}} \quad (1 \leq \mu \leq d) \quad (2)$$

The fermionic part of the action is a bilinear of the fermion fields, $S_F = \bar{\chi} \Delta \chi$, with $\Delta = mI + \Lambda$, and Λ an antihermitian matrix

$$\Lambda_{x,y} = \frac{1}{2} \sum_{\mu} \eta_\mu(x) [\delta_{x+\mu,y} U_\mu(x) - \delta_{x-\mu,y} U_\mu^*(x - \mu)] \quad (3)$$

$$1 \leq \mu \leq d$$

Due to its bilinear nature, we can integrate analytically the fermionic part of the action in the partition function, thus obtaining

$$Z = \int [d\bar{\chi}] [d\chi] [dU] e^{-\beta S_G - \bar{\chi} \Delta \chi} = \int [dU] \det \Delta e^{-\beta S_G} = \int [dU] e^{-S_{eff}} \quad (4)$$

with the effective action defined as

$$S_{eff} = \beta S_G - \ln \det \Delta = \beta S_G - \text{tr} \ln \Delta \quad (5)$$

Now we are going to consider the loop expansion of the effective action, as well as the expansion of the determinant of the Dirac operator.

A. The hopping parameter expansion

Let us examine first the expansion of the effective action [8]. We have

$$\begin{aligned} \text{tr} \ln \Delta &= \text{tr} \ln(mI + \Lambda) = \text{tr} \ln(m(I + m^{-1}\Lambda)) \\ &= V \ln m + \sum_{k \geq 1} (-1)^{k+1} \frac{m^{-k}}{k} \text{tr}(\Lambda^k) \end{aligned} \quad (6)$$

The elements of Λ have a natural interpretation as directed links on the lattice: due to the local character of the action, only elements corresponding to pairs of nearest-neighborhood sites are non zero, so we can assign to each non-zero element $\Lambda_{x,y}$ the unique directed link which goes from x to y . We can thus rewrite the previous traces as

$$\begin{aligned} \text{tr}(\Lambda^k) &= \sum_{x, \mu_1, \mu_2, \dots, \mu_k} \Lambda_{x, x+\mu_1} \Lambda_{x+\mu_1, x+\mu_1+\mu_2} \dots \\ &\dots \Lambda_{x+\mu_1+\dots+\mu_{k-1}, x+\mu_1+\dots+\mu_{k-1}+\mu_k} \delta_{\mu_1+\dots+\mu_k, 0} \end{aligned} \quad (7)$$

In this paragraph only, and for the sake of clarity, the μ_i represent backward as well as forward unit vectors in the lattice.

We can assign to each non-vanishing term of the trace of Λ^k a unique closed and oriented loop on the lattice constructed with k links. Since loops must be closed, it is obvious that k has to be even in order to give a non-zero contribution to (6). Consequently we rewrite the trace as

$$\text{tr} \ln \Delta = V \ln m - \sum_{k \geq 1} \frac{m^{-2k}}{2k} \sum_{l \in \mathcal{G}_{2k}} \Lambda_l \quad (8)$$

where \mathcal{G}_j represents all the loops with j links in the class considered, and Λ_l is a notation for the oriented product of the elements of Λ around loop l . The net result is real, as for the contribution of each loop we have the conjugate contribution of the loop taken in the opposite direction. To examine more closely this expression, and specifically the sign of the different terms, we use the definition of Λ . For each backward link we get a minus sign. But for a closed loop, exactly half of the links are backward links, so for a loop with $2k$ links we get a factor of $(-1)^k$. On the other hand we have to take into account the Kogut-Susskind phases. One simple way to do this is to absorb them into the links, with a trivial change of variables, $W_\mu(x) = \eta_\mu(x) U_\mu(x)$. The corresponding change induced in the plaquettes consists in multiplying each of them by -1 . The final result for the effective fermion action expressed in the new variables $W_\mu(x)$ is

$$S_{eff}^F = -V \ln m + \sum_{k \geq 1} (-1)^k \frac{m^{-2k}}{2k(2)^{2k}} \sum_{l \in \mathcal{G}_{2k}} W_l \quad (9)$$

where W_l means the directed product of the W variables along loop l . We can see from this expression that the effective action is frustrated in the sense that there is no single configuration that minimizes term by term the expansion. Different terms are minimized by different configurations. It should be said that even if we have used Kogut-Susskind fermions in our derivation, the conclusion remains valid for Wilson fermions.

It is interesting to note that the analogous derivation for a scalar theory coupled to an Abelian gauge field does not show any kind of frustration. The essential differences with respect to the fermionic case are on one hand that the corresponding determinant appears in the denominator, and on the other hand that in the scalar case the Kogut-Susskind phases are absent. The final result of the development of the effective action for the scalar case is the following:

$$S_{eff} = \beta S_G + V \ln m - \sum_{k \geq 1} \frac{m^{-2k}}{2k(2)^{2k}} \sum_{l \in \mathcal{G}_{2k}} U_l \quad (10)$$

In this case each term of the expansion is minimized by the trivial configuration, so we have no frustration.

B. The expansion of the determinant

The expansion discussed in the previous section is the most convenient approach for massive fermions. However one can always proceed directly with the expansion of the determinant of Δ , as long as the volume is kept finite [9]. So let us consider

$$\det \Delta = \det(mI + \Lambda) = \sum_{i_1, i_2, \dots, i_V} \epsilon_{i_1, i_2, \dots, i_V} \Delta_{1, i_1} \cdots \Delta_{V, i_V} \quad (11)$$

where $\epsilon_{i_1, i_2, \dots, i_V}$ is the totally antisymmetric symbol in V indices. Due to gauge invariance and to the properties of the determinant, each non-vanishing contribution to (11) can be associated to a graph on the L^d lattice made up of some number of isolated sites (each one contributing with a factor m) and some number of closed, directed, simple (i.e., no link can appear twice) and non-intersecting loops. In other words, after eliminating the isolated sites, all the rest of the lattice should be covered with closed loops in such a way that in each site there are exactly one incoming and one outgoing link. In particular this implies that the number of isolated sites (and so the powers of m) should be even.

To minimize the contribution of the fermionic part to the effective action (5) one should look for configurations that maximize the fermionic determinant. In general, it is impossible to maximize simultaneously with a single configuration all the terms in the expansion of the determinant (this can be seen easily examining simple graphs containing two isolated sites, for example).

III. THE TOY MODEL

Motivated by the expansions analyzed in the previous section, we have introduced an Abelian gauge toy model with explicit frustration [7]. We have studied the effect of frustration in this model as a first step towards the analysis of the effect of frustration in more realistic models, like QED. The model we have studied is a $U(1)$ pure gauge model defined in a hypercubic lattice by the following action:

$$S = S_4 + S_6 = -\beta \sum_{pl} \text{Re} U_{pl} + \beta_6 \sum_{p_6} \text{Re} U_{p_6} \quad (12)$$

with $\beta, \beta_6 \geq 0$. The first part (S_4) is the standard compact $U(1)$ pure gauge Wilson action. The second part (S_6) is defined as the sum of contributions over all closed loops made up with six non-repeated links. These loops can be generally classified in three different classes: planar loops, loops that involve two planes and loops that involve three planes (obviously in the special case $d = 2$ only planar loops appear).

A. Classical ground states and order parameter

It is not difficult to realize the existence of frustration in this model at the classical level in $d > 2$, even if we consider only the S_6 piece of action (12). In fact, it is not possible to minimize simultaneously the contributions to the action of all the loops. One could try to work out the generic minima for some of the classes of loops but instead, we will describe here, motivated by our numerical results, a specific class of configurations that appear to be the classical ground states accessible to the system. This will permit us to define a natural order parameter for this system.

In the configurations considered the plaquettes take the values ± 1 . It is easy to see that we can find configurations of this type which are minima for the planar and three-plane loops simultaneously. For the planar loops, the minimum condition requires that the sign of the plaquette should be alternate on every plane (for this reason we will call these configurations antiferromagnetic, in analogy with magnetic systems). Among these configurations we can find a subset that minimize also the three-plane loops. There are a total of $2^{(d-1)}$ configurations of this type, related by simple symmetry transformations. Now we should consider the two-plane loops. It is easy to check that these loops are not simultaneously minimized in the configurations that we have described. In fact, the net contribution to the action of these loops is zero in any of the minima considered. In the case $d = 2$ only the planar loops appear in the action; therefore the antiferromagnetic configurations are global minima and the system is not frustrated.

On the other hand the minimum for the S_4 part of the action is reached for the homogeneous configuration $U_{pl} = 1$, which becomes another source of frustration when considering the complete system.

This analysis suggests the introduction of an order parameter for each set of parallel planes, the staggered plaquette, defined as follows:

$$P_{\mu\nu}^s = \frac{1}{V} \sum_x \epsilon(x) \text{Re} U_{\mu\nu}(x), \quad \epsilon(x) = (-1)^{x_1 + x_2 + \dots + x_d} \quad (13)$$

This order parameter is different from zero in the antiferromagnetic vacuum and vanishes in the ferromagnetic (homogeneous) one. For the configurations described before it takes the values ± 1 .

B. The case $d = 2$ and the XY model

We will show in this subsection how, in $d = 2$, the thermodynamic limit of our model is the XY model.

To this end let us consider first the S_6 part of action (12). In two dimensions it contains only planar loops. It is a well known fact that in $d = 2$ it is possible to replace

the links by the elementary plaquettes as independent variables of the theory. This is due to the fact that there are no cubes in the lattice, and hence only one Bianchi identity: the product of all plaquettes must be equal to 1. Therefore the number of independent plaquettes is $V - 1$. If we consider all the plaquettes as independent variables, the difference in the action is a term of order $O(1)$, which is irrelevant in the large V limit. The plaquette variables can now be assigned to the sites of the dual lattice, with an action which involves only nearest neighborhoods. Now, to obtain the usual form for the XY model, we still need another change of variables. We consider the usual bipartition of the planar lattice, and we change all the plaquettes of one of the sublattices to minus their complex conjugate. If we do this, the final form for the action is

$$S_6 = \beta_6 \sum_{\langle i,j \rangle} \text{Re}(U_i(-U_j^*)) = -\beta_6 \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \quad (14)$$

where $\sum_{\langle i,j \rangle}$ means a sum over all pairs of nearest neighborhood sites of the (dual) lattice and θ_k is the angle corresponding to the element of $U(1)$ in each site. But this is just the action of the XY model on a square lattice. As is well known, this model has only one ground state and does not present frustration.

The contribution of the S_4 term to (12), expressed in the new variables, can be written as

$$S_4 = -\beta \sum_i \text{Re}(U_i) = -\beta \sum_i \epsilon_i \cos(\theta_i) \quad (15)$$

where ϵ_i equals ± 1 according to which sublattice the site i belongs to. We see that the full action for the two-dimensional model,

$$S = -\beta_6 \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) - \beta \sum_i \epsilon_i \cos(\theta_i) \quad (16)$$

is the action of the X-Y model in the presence of an external staggered magnetic field. The X-Y model with an external random field has been extensively studied ([10,11] and references therein) as a simple example of frustrated system, but we are not aware of studies of the staggered case, so we do not know to what extent frustration is relevant for this system.

IV. NUMERICAL SIMULATIONS

We performed Montecarlo simulations of this model in $d = 3$ and $d = 4$ using a standard Metropolis algorithm. The observables which were measured in the simulations were the staggered plaquette and the usual normalized plaquette in each plane, and the normalized contribution of the 6-loop part of the action,

$$P_{\mu\nu}^s = \frac{1}{V} \sum_x \epsilon(x) \text{Re} U_{\mu\nu}(x), \epsilon(x) = (-1)^{x_1+x_2+\dots+x_d} \quad (17)$$

$$1 \leq \mu < \nu \leq d$$

$$P_{\mu\nu} = \frac{1}{V} \sum_x \text{Re} U_{\mu\nu} \quad (18)$$

$$P_6 = \frac{1}{\mathcal{N}V} \sum_{p_6} \text{Re} U_{p_6} \quad (19)$$

where \mathcal{N} means the number of 6-loops per volume. We also measured for completeness the imaginary part of these quantities, that gave results consistent with zero in all our simulations.

In order to obtain a qualitative understanding of the phase diagram of this model, our strategy was to explore the parameter space over several lines of constant β . To locate the possible transition points we did a number of annealing cycles over β_6 , while maintaining β fixed to a given value. The most extensive simulations were done in $d = 4$, and we discuss this case in more detail below.

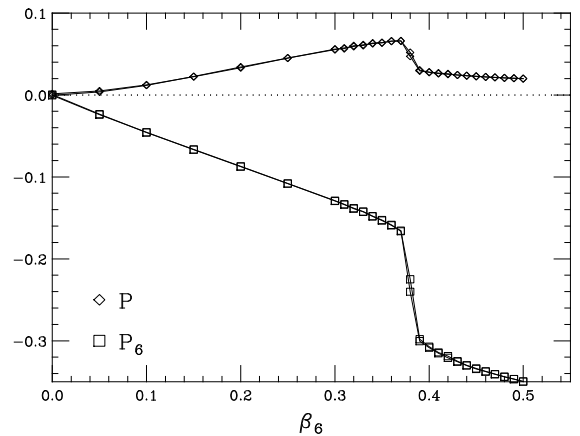


Fig. 1

FIG. 1. Plaquette and 6-loop hysteresis cycles at $\beta = 0$ against β_6

As we have shown before, in $d = 2$ our model is equivalent to the X-Y model. It is well known that this model presents a continuous transition at a value of the coupling constant $\beta_6 \simeq 0.9$, at least for $\beta = 0$.

In Fig. 1 and 2 we show the results for the annealing cycles in $d = 3$ at $\beta = 0$. The figures suggest the existence of a transition into a phase with a non-vanishing value of the staggered plaquette at a smaller value of the coupling constant than in $d = 2$, $\beta_6 \simeq 0.38$. However, a more careful study and larger statistics are necessary to establish unambiguously the order of the transition.

A. The system in $d = 4$

We simulated the four-dimensional system along several lines of constant $\beta \geq 0$. First, we did several runs at

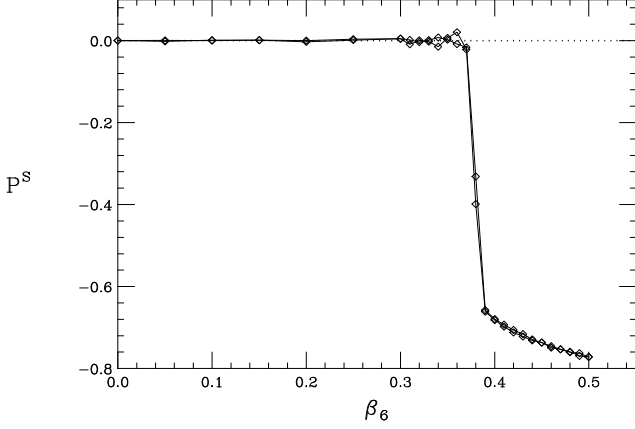


Fig. 2

FIG. 2. Staggered Plaquette hysteresis cycle at $\beta = 0$ against β_6

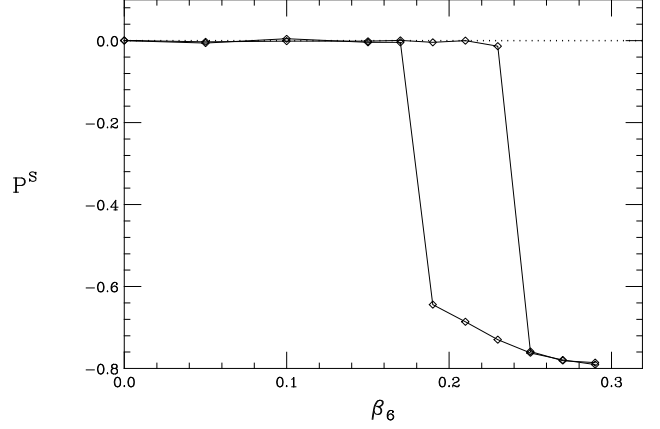


Fig. 4

FIG. 4. Staggered Plaquette hysteresis cycle at $\beta = 0$ against β_6

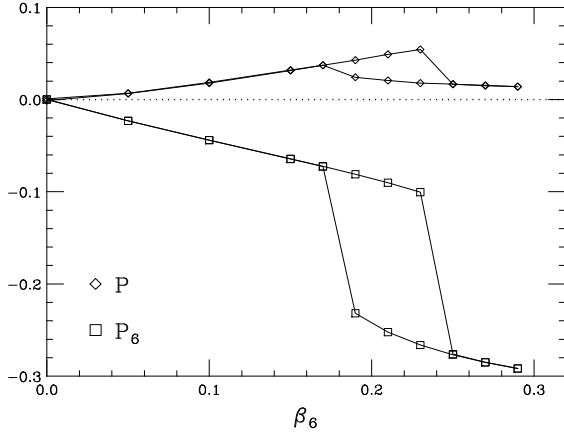


Fig. 3

FIG. 3. Plaquette and 6-loop hysteresis cycles at $\beta = 0$ against β_6

different β , while maintaining β_6 fixed to zero, in order to obtain properly thermalized configurations that we used later to start the annealing cycles in β_6 . For each value of β , we started the annealing with the thermalized configuration obtained in this way. After we finished the simulation for a given value of β_6 , we changed it by some small amount and repeated the process starting with the last configuration generated at the previous value of β_6 . Proceeding in this way we did a complete forward-backward cycle. These simulations were done in 4^4 and 6^4 lattices.

We show in Fig. 3 and 4 the resulting annealing cycles for some of the magnitudes we have measured at $\beta = 0$. The staggered plaquette is shown here only for one of the planes (see the discussion below). We see a very clear hysteresis effect signaling a strong first-order phase transition. In order to check that this is not the

effect of a bad thermalization around a continuous transition we compared cycles obtained by increasing the simulation time over two orders of magnitude, from 1000 to 100000 Montecarlo sweeps, and no change in the width of the hysteresis region was seen. Also no significant finite-size effects were found in our simulations. We also implemented an over-relaxation procedure which was combined with the Metropolis algorithm; again, no change in the measured magnitudes was observed. Concerning the nature of the large β_6 phase, we can see in figure 4 that the staggered plaquette shows a sudden increase from zero, when entering into this phase. If we compare its value for the different planes we see that it only differs in sign, with the signs being consistent with the ones obtained from the analysis of the classical ground state configurations done in section (III A). The simulations at larger values of β_6 , produced configurations which were essentially the classical ground states described before.

Now let us discuss our results for the lines at constant $\beta > 0$. For small values of β , the results of the simulations are qualitatively the same as for $\beta = 0$, apart from the fact that the transition point occurs at higher values of β_6 for higher values of β . But we know that this system, for $\beta_6 = 0$, has a first-order transition at $\beta \simeq 1.0$, the usual confined-deconfined transition for the standard compact $U(1)$ pure gauge theory [12–14]. So it is very likely that a line of transition points emerges from the β axis and, indeed, this is what we found. We did simulations at several values of β above the transition point. In Fig. 5 and 6 we show the results obtained by cycling over β_6 while keeping $\beta = 1.5$. We can clearly see here two transitions: the first one should correspond to the continuation of the usual confined-deconfined transition (the staggered plaquette remains zero for this transition), whereas the second one corresponds to the transition to the antiferromagnetic phase, as clearly shown by the jump of the staggered plaquette. The results for all

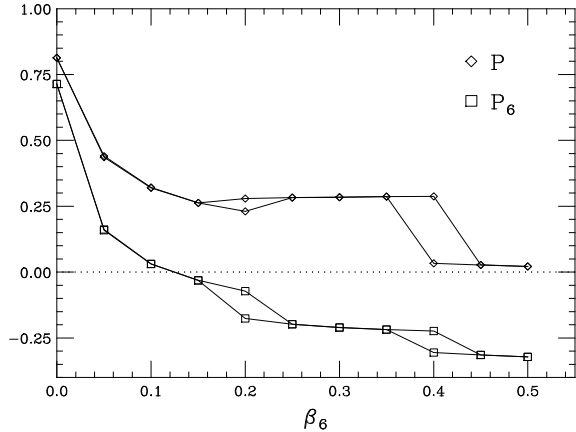


Fig. 5

FIG. 5. Plaquette and 6-loop hysteresis cycles at $\beta = 1.5$ against β_6

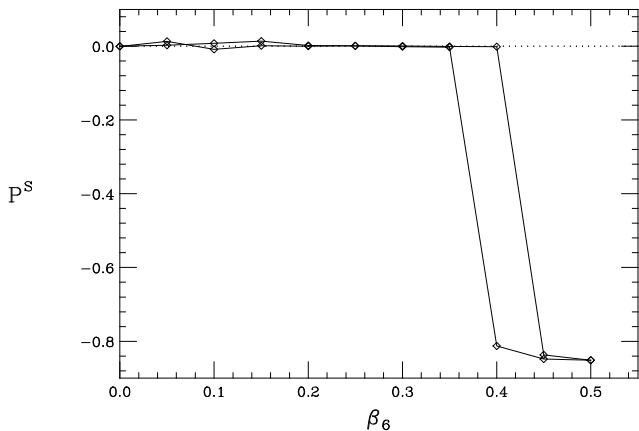


Fig. 6

FIG. 6. Staggered Plaquette hysteresis cycle at $\beta = 1.5$ against β_6

the β values are consistent with this picture, and permit us to obtain a qualitative phase diagram, that we show in Fig. 7. We found three phases separated by two first-order lines. The antiferromagnetic phase is characterized by a non-zero value of the staggered plaquette, which is zero in the other two phases. In this phase we have several states, to be precise eight states, related by a spontaneously broken translational symmetry. The other two phases are the continuation of the confined and unconfined phases of the standard compact pure gauge model.

V. COMMENTS

Our starting point was the observation that competing interactions are present in the effective gauge theory

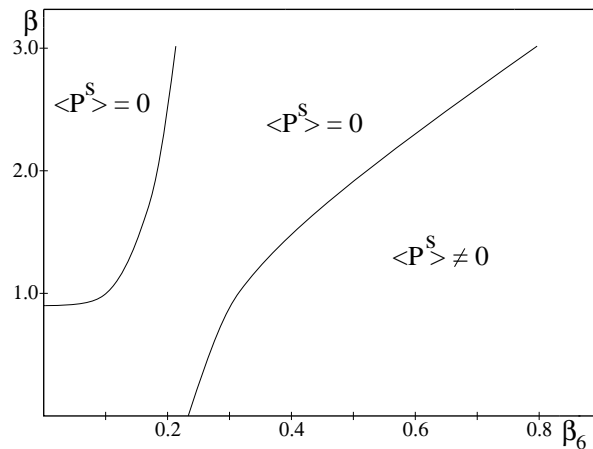


FIG. 7. Phase diagram

obtained, after integration on Grassmann fields, from a gauge-fermion theory on the lattice, such as *QED*. Inspired from analogies with condensed matter models like spin-glasses in which frustration and disorder conspire to give non standard phases to the system, we asked if the rich, and in some sense unexpected phenomenology observed in *3d* and *4d* lattice *QED* can be ascribed to the presence of non standard vacua.

The first step in our analysis has been the study of a simpler model, in which frustration is introduced by hand, but as suggested by the hopping parameter expansion. In the $d = 3$ and $d = 4$ cases we have seen clear signals of the existence of an antiferromagnetic phase in which translational invariance is broken. This is not totally surprising since the solutions of classical equations of motion, at least in a limiting case of the parameter space, share this property. A crucial point in the analysis has been the availability of a clear and simple order parameter, inspired from the properties of these classical configurations. The $d = 2$ case was found analytically equivalent to the $X - Y$ model with staggered external magnetic field in the thermodynamical limit.

The simple model we have considered has only a finite number of competing interactions, so on general grounds we cannot expect to find spin glass-like phases. On the contrary, in the full model, the number of frustrated terms in the effective action is diverging with the volume of the system. We do not know if this is enough to simulate in some way the inclusion of disorder, that in a spin glass system is the other ingredient for the appearance of non standard phases: this is an issue for future work. The first problem we have to face is the definition of a suitable order parameter.

The list of features in three and four dimensional *QED* that lack an explanation is long, and we suspect that, at least some of them, could be related to frustration at the level of the effective action.

ACKNOWLEDGMENTS

This work has been partly supported through a CICYT (Spain) - INFN (Italy) collaboration.

E. Follana acknowledges support from Caja de Ahorros de La Inmaculada.

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